

X, Y be top. spaces

Def $\pi: X \rightarrow Y$ is a covering map if $\forall y \in Y$,
 $\exists V \ni y$ open s.t. $\pi|_U: U \rightarrow V$ is homeo for each
 component U of $\pi^{-1}(V)$

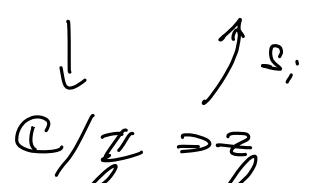
Lemma $|f^{-1}(y)|$ is locally constant

Automorphisms of $\pi: X \xrightarrow{\sim} X$, $\text{Aut}(\pi)$

Ex: $z \mapsto z^n$ on $\mathbb{D}^1$, $z \mapsto z^n$ on $\mathbb{D}^1$

Ex $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{Z}$

For each $z \in \mathbb{C}/\mathbb{Z}$, pick any $U \ni z$ s.t. a
 sector $S = \pi^{-1}|_U$ exist. Any two sectors differ by:
 $S_1(z) = S_2(z) + n$, so the transition map $\mathbb{C} \ni z+n$
 is homeo.



Simply-connected : every simple closed
 curve $\gamma \subset X$ bounds a disk

Prop. $\exists! \tilde{X}$ simply-connected, $\pi: \tilde{X} \rightarrow X$

Univ. covering, and $\tilde{X} \xrightarrow{\pi} X$
 $\tilde{X} \xrightarrow{\sim} \tilde{X}/\text{Aut}(\pi)$

Lifting prop: $\tilde{X} \xrightarrow{\pi} X$
 $\tilde{X} \xrightarrow{\sim} \tilde{Y}$
 $\downarrow \quad \downarrow$
 $X \xrightarrow{f} Y$

Prop. $\Gamma \subset \text{Home}(X)$ act

1) properly discontinuous : every compact $K \subset X$

only intersect finitely many $y \cdot x \neq x \forall y \neq 1$
 2) finitely $y \cdot x \neq x \forall y \neq 1$

Then $X \rightarrow X/\Gamma$ is a covering map

Prop. $f: X \rightarrow Y$ is covering, Y is a R.S. $\rightarrow X$ is a R.S. so that f is a homeomorphism.

— Deck transf. are homeomorphisms: $X \xrightarrow{\sim} X$
 $\downarrow \gamma$

Thm. (Unif.) If X is s.c., then
 $X \cong \mathbb{C}, \hat{\mathbb{C}}, \text{ or } \mathbb{D}.$

Cor. If X is a R.S., then $X \cong \mathbb{C}/\Gamma, \hat{\mathbb{C}}/\Gamma, \mathbb{D}/\Gamma.$
Euclidean $\rightarrow$ Spherical Hyperbolic

Spherical: No free actions, so $X \cong \hat{\mathbb{C}}.$

Euclidean: $z \mapsto az+b$ fix pt if $a \neq 1.$

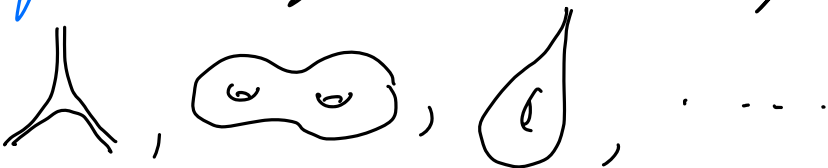
$\rightarrow$ Free + proper action $\Rightarrow \Gamma \subset \mathbb{C}$ discrete

$$\Gamma = \{0\} : \mathbb{C}$$

$$\Gamma = \mathbb{Z}\langle a \rangle : \mathbb{C}/a\mathbb{Z} \text{ Cylinders } \boxed{} \text{ rank } -1$$

$$\Gamma = \mathbb{Z}\langle a, b \rangle : \mathbb{C}/\Gamma \text{ Elliptic curve / torus } \odot \text{ rank } -2$$

Hyp: Everything else: in particular, if $\chi < 0$:



Prop. X is a R.S. $\forall$ $\chi < 0$. All maps $X \rightarrow Y$

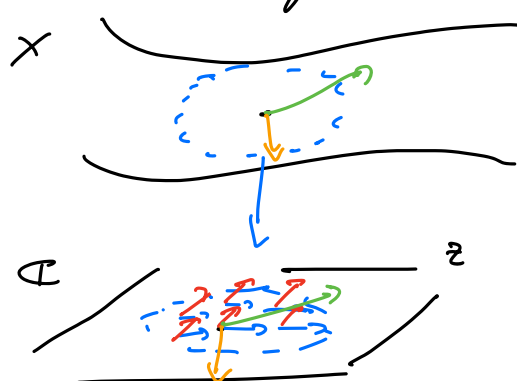
$\hat{\mathbb{C}} \rightarrow X, \hat{\mathbb{C}} \rightarrow Y$ are constant.

Cor. (Picard) If a hsb map $\mathbb{C} \rightarrow Y$ hits two points, then it is constant.

Conformal metrics:

R.S. have a notion of angle, but not length

Def. (Riemannian metric) A smoothly varying family $g_z(\cdot, \cdot)$ of real inner products on the tangent space T_z^X



else a basis:

$$g_z(v(z), w(z)) = \frac{1}{w(z)} \begin{pmatrix} a(z) & b(z) \\ b(z) & c(z) \end{pmatrix} \begin{pmatrix} v(z) \\ w(z) \end{pmatrix}$$

v. fields $v(z), w(z)$

A metric g on R.S. X is conformal if the angles measured using the R.S. charts are the same as the charts measured using g .

Goal: Find any R.S. (usually canonical) complete + conf. metric. ex: $\mathbb{D} \subset \mathbb{C}$ - Euclidean metric gives rise to all time

Euclidean metric.

$$\mathbb{C} : z = x + iy, \quad g_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

Constant curvature 0, $\text{Aut}(\mathbb{C})$ are not all isometries,

$$\text{Isom}^+(\mathbb{C}) = \{ z \mapsto e^{i\theta} z + c \}$$

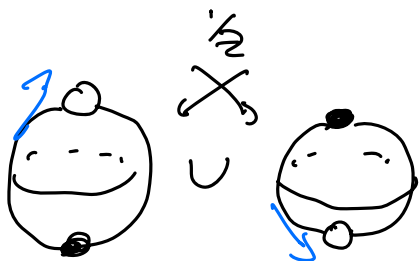
Spherical metric

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Spherical metric

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \quad z = x + iy, \quad g_z = \begin{pmatrix} \frac{4}{(1+|z|^2)^2} & 0 \\ 0 & \frac{4}{(1+|z|^2)^2} \end{pmatrix}$$

Laurent series $f(z) = \frac{1}{z} : \quad \frac{w.r.s:}{g_{f(z)}(f(z) \cdot v, f'(z) \cdot w)} = g_z(v, w)$
 $f'(z) = -\frac{1}{z^2}$



$$g_{f(z)} = \frac{4|z|^4}{(1+|z|^2)^2} I$$

trans pose is ca_j :

$$\begin{aligned} g_{f(z)}(f'(z) \cdot v, f'(z) \cdot w) &= \left(\frac{-1}{\bar{z}^2} w \right)^T \begin{pmatrix} 4|z|^4 \\ (1+4|z|^2)^2 \end{pmatrix} \left(\frac{-1}{\bar{z}^2} v \right) \quad \begin{matrix} \bar{z} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \\ z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{matrix} \\ &= w^T \left(\frac{-1}{\bar{z}^2} \right) \begin{pmatrix} 4|z|^2 \\ (1+|z|^2)^2 \end{pmatrix} \left(\frac{-1}{\bar{z}^2} \right)^T v \\ &= w^T \cdot \frac{4|z|^2}{(1+|z|^2)^2} v \end{aligned}$$

• Constant counter = ∞

Isometry group: $SO(3) \subset PSL(2, \mathbb{C})$

Hyperbolic metric

$$A1: g_z = \begin{pmatrix} \text{Im}(z)^{-2} & 0 \\ 0 & \text{Im}(z)^{-2} \end{pmatrix}$$

$$z = x + iy$$

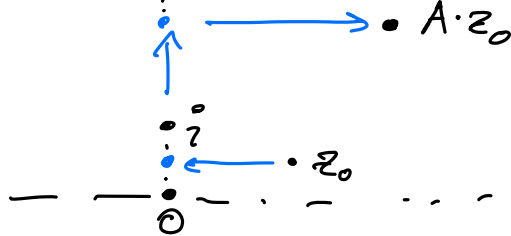
Claim. $Isom^+ H = PSL(2, \mathbb{R}) = \mathcal{AH}(H)$

1. $z \mapsto z + r, r \in \mathbb{R}$ is an isometry

2. $z \mapsto kz$, $k \in \mathbb{R}$ is continuity:

$$g_{k_z}(k \cdot v, k \cdot \omega) = (k_\omega)^\perp \mathcal{I}_m(k_z)^{-2} \mathcal{I}(k_v) = \omega^\perp \mathcal{I}_m(z)^{-2} \mathcal{I}_v = g_z(v, \omega)$$

$$\exists A \in \text{Iso}^r H \quad \exists B \in \text{Iso}^r H$$



$$B \cdot z_0 = A \cdot z_0$$

$$\Rightarrow B^{-1}A \in \text{PSL}_2 \mathbb{R} \text{ fixes } z_0$$

$$\mathbb{H} \xrightarrow{B^{-1}A} \mathbb{H}$$

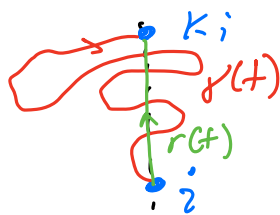
$$\cong \uparrow \quad \downarrow \cong$$

$$\mathbb{D} \xrightarrow{\quad} \mathbb{D} \text{ fixes } \text{angpt.}$$

Schwarz lemma $\Rightarrow$ Rotator $\Rightarrow |(B^{-1}A)'(z)| = 1$, $B^{-1}A$ is an isometry at z_0 , so A is at z_0 .

Geodesics:

Project $\gamma(t) \pm r(t)$

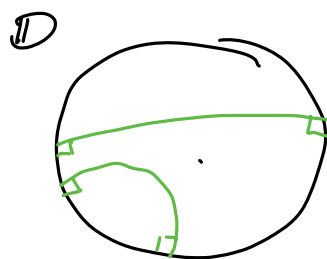
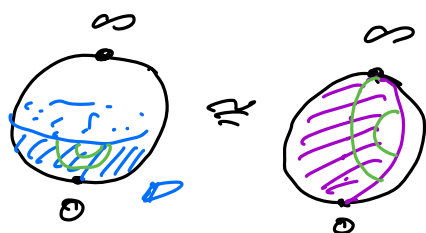


$$l_{\text{hyp}}(\gamma) = \int_0^1 |\gamma'(t)|_{\text{hyp}} dt = \int_0^1 \frac{|\gamma'(t)|}{\text{Im } \gamma(t)} dt$$

$$\geq \int_0^1 \frac{|r'(t)|_{\mathbb{H}}}{r(t)} dt = \int_1^K \frac{dt}{t}$$

$$= \log K$$

- Möbiel send circles to circles on $\mathbb{C}$, so geodesics are sent. In $\mathbb{R}$ semicircle $\perp$ to $\mathbb{R}$.
- Hyp. Möbiel is complete (geodesic split for all time) and conformal, counter-1.
- Poincaré disk model:



+ Hyp. metrics on all hyp. Riemann surfaces.

Thm. (Pick) $f: X \rightarrow Y$ is mob, $X \neq Y$ are hyp. Riemann surfaces, then either:

1) $f: X \xrightarrow{\sim} Y$ is an isomorphism and isometry

2) f is a covering map but not 1-1, and
then f is locally an isometry

3) f is contracting

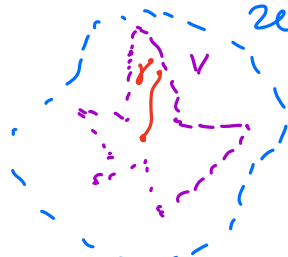
Ex: 1) $\text{Aut } H = \text{Isom}^+ H$

2)
$$\begin{array}{ccc} w \in H & \xrightarrow{w \mapsto 2w} & H \\ \downarrow c_w & & \downarrow \\ D - \{0\} & \xrightarrow{z \mapsto z^2} & D - \{0\} \end{array}$$
 hyp. isometry
covering map

3) $H \rightarrow H$ contract distances
 $z \mapsto z + i$

Proof.
$$\begin{array}{ccc} H & \xrightarrow{\tilde{f}} & H \\ \downarrow \text{isometries} & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$
 $\tilde{x} \mapsto \tilde{y}$
 $\downarrow$
 $x \mapsto y = f(x)$ $|D\tilde{f}_{\tilde{x}}|_{\text{hyp}} = |Df_x|_{\text{hyp}}$

Pick $A \in \text{Isom}^+ H$, $A \cdot \tilde{y} = \tilde{x}$. By Schwarz lemma, $|D\tilde{f}_{\tilde{x}}| = |D(A \cdot \tilde{f})_{\tilde{x}}| \leq 1$
w/ equality iff $\tilde{f}$ is an isomorphism.

Ex:  $v \subsetneq u$ is strictly contracting:
$$l_v(y) > l_u(y).$$